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Graded contractions and bicrossproduct structure of deformed inhomogeneous algebras

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Abstract. A family of deformed Hopf algebras corresponding to the classical maximal isometry algebras of zero-curvature N-dimensional spaces (the inhomogeneous algebras $iso(p, q)$, $p + q = N$, as well as some of their contractions) are shown to have a bicrossproduct structure. This is done for both the algebra and, in a low-dimensional example, for the (dual) group aspects of the deformation.

1. Introduction

The procedure to deform simple algebras and groups was established by Drinfel'd [1], Jimbo [2] and Faddeev *et al* [3]. The algorithm, which leads to the so-called 'quantum' algebras, does not cover, however, the case of non-semisimple algebras. Since the contraction process leads to inhomogeneous algebras by starting from simple ones, it is natural to use it as a way to deform inhomogeneous Lie (i.e. 'classical' or undeformed) algebras. This path of extending the classical idea of the Lie algebra contraction to the case of deformed algebras was proposed by Celeghini *et al* [4]. The basic requirement to define a deformed inhomogeneous algebra is the commutativity of the processes of contraction and deformation: when considering a simple algebra and one of their inhomogeneous contractions, both at classical and deformed levels, the deformation of the contracted inhomogeneous Lie algebras should coincide with the contraction of the deformed simple algebra. This commutativity is not always guaranteed, and in general requires [4] a redefinition of the deformation parameter q in terms of the contraction parameter and the new deformation one, so that q is not a passive element in the contraction. This was used, for instance, to obtain the κ-Poincaré algebra [5], for which the deformation parameter κ has dimensions of inverse length.

The concept of contraction of Lie algebras (or groups) was discussed in the early 1950s by İnönü and Wigner [6] (see also [7]). The idea of group contraction itself arose in the group

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analysis of the non-relativistic limit, and its applications to mathematical physics problems have been very fruitful. The study of details behind this procedure unveils interesting mathematical structures, which in many important cases are linked to physical properties. In particular, the contraction process may increase the group cohomology [8] (see also [9]), as is the case in the standard non-relativistic limit. Several attempts have been made to systematize the study of contractions and recently a new approach has been put forward in [10], under the name of graded contractions. The key idea there is to preserve a given grading of the original Lie algebra. This condition may fit neatly with physical requirements and is automatically satisfied in the simplest case of the İnönü–Wigner contractions, which correspond to the simplest $\mathbb{Z}_2$-grading.

A class of Lie algebras describing a whole family of contractions is the so-called orthogonal Cayley–Klein (CK) algebras. The name is due to historical reasons: these are the Lie algebras of the motion groups of real spaces with a projective metric [11] (see also [12]). The same family appears as a natural subset of all $\mathbb{Z}_2^{\otimes N}$-graded contractions which can be obtained from $so(N+1)$ [13]. And furthermore, among orthogonal CK algebras we find not only all simple pseudo-orthogonal algebras, but many non-semisimple algebras of physical importance, such as the kinematical Poincaré and Galilei algebras in $(N-1, 1)$ dimensions, the Euclidean algebra in N dimensions, etc. The CK scheme does not deal with a single Lie algebra, but with a whole family of them simultaneously, each of which is parametrized by a set of real numbers with a well defined geometrical and physical significance. The main point to be stressed is the ability of this kind of approach to describe some properties of many Lie algebras in a single unified form. This is possible as the Lie algebras in the CK family, though not simple, are 'very near' to the simple ones, and many structural properties of the simple algebras, when suitably reformulated, still survive for the CK algebras.

It is possible to give deformations of algebras in the CK family; naturally enough these will be said to belong to the CK family of Hopf 'quantum' algebras. In [14] deformations of the enveloping algebras of all algebras in the CK family of $so(p, q)$, $p+q=3, 4$ were given. For higher dimensions, i.e. for algebras in the family of $so(p, q)$, $(p+q=N+1)$ with $N>3$, a quantum deformation of the general parent member of the CK family is still not known, yet there exists a scheme of quantum deformations encompassing all motion algebras of flat affine spaces in N dimensions, which include the ordinary inhomogeneous $iso(p, q)$, $(p+q=N)$ [15]. This scheme provides a Hopf algebra deformation for each algebra in the family. Some of its members are physically relevant non-semisimple algebras, and include as particular cases most of the deformations of these algebras found in literature.

An important fact in quantum algebra/group theory is the (co)existence of two closely linked algebraic structures: the algebra (as expressed by the commutators or the commuting properties of the algebra of functions on the group) and the coalgebra (as given by the coproduct). Most of the complications found when doing quantum contractions can be traced to the need to deal simultaneously with these two aspects. For instance, a naive contraction might lead to divergences either in the coproduct or in the R-matrix [4, 16]. One of the main motivations behind the CK scheme was to be able to describe at the same time a family of algebras, including some simple and some contracted algebras, in such a way that the possible origin of divergences under contractions is clearly seen and controlled.

In this paper we address a specific problem where the advantages of a CK-type scheme are exhibited. In the classical case, an İnönü–Wigner (IW) contraction of a simple algebra leads to a non-semisimple one which is the semidirect sum of an Abelian algebra and the preserved subalgebra of the original algebra with respect to the contraction was made. *All* IW contractions of simple algebras have a semidirect structure. It is then natural to ask: is there a similar pattern for the contracted deformations, i.e. for the Hopf algebra deformations of contracted simple Lie algebras? The analogue of the semidirect product is an example of the bicrossproduct of Hopf algebras, introduced by Majid [17] (see also [18, 19]). The aim of this paper is to show that all deformed algebras in the affine† CK family $iso_{\omega_2,\dots,\omega_N}(N)$ have indeed a bicrossproduct structure, as is the case of the κ-Poincaré [21]. This result opens the possibility of recovering more easily the deformed dual groups $\mathrm{Fun}_q(ISO_{\omega_2,\dots,\omega_N}(N))$ by using the dual bicrossproduct 'group-like' expressions (see [20] for some group-like (rather than algebra-like) examples of this construction). Classically, the $iso_{\omega_2,\dots,\omega_N}(N)$ family includes all inhomogeneous Lie algebras $iso(p,q)$ $(p+q=N)$, so we will refer loosely to the aim of the paper as showing the bicrossproduct structure of deformed inhomogeneous groups. It should be kept in mind, however, that we are referring to a specific deformation, and that examples exist (see [20]) where a contraction of a deformed algebra has no bicrossproduct structure.

The paper is organized as follows. In section 2 we briefly describe the classical CK algebras and present a discussion on contractions and dimensional analysis since this is relevant for the assignment of physical dimensions to the deformation parameters. In section 3 we give the explicit expressions for their q-deformations. The bicrossproduct structure of these q-deformed CK Hopf algebras is shown in section 4. Examples of this structure for physically interesting algebras are presented in section 5. In section 6 we show, as an example, how to obtain the (dual) group deformation in the case of lowest dimension $N=2$. In section 7 we present our conclusions and we close the paper with an appendix.

2. Affine CK Lie algebras and dimensional analysis

2.1. The CK scheme of geometries and Lie algebras

The complete family of the $so(N+1)$ CK algebras is a set of real Lie algebras of dimension $(N+1)N/2$, characterized by N real parameters $(\omega_1,\omega_2,\dots,\omega_N)$ [12]. This family appears, for example, as a natural subfamily [13] of all the graded contractions from the Lie algebra $so(N+1)$ [22] corresponding to a $\mathbb{Z}_2^{\otimes N}$ grading of $so(N+1)$, and its elements will be denoted $so_{\omega_1,\omega_2,\dots,\omega_N}(N+1)$; in particular, $so_{1,1,\dots,1}(N+1)\equiv so(N+1)$. In terms of a basis of $so_{\omega_1,\omega_2,\dots,\omega_N}(N+1)$ adapted to the grading, $\{\mathbb{J}_{ab};\ a<b,\ a,b=0,1,\dots,N\}$, this family of algebras is defined by

$$[\mathbb{J}_{ab},\mathbb{J}_{ac}]=\omega_{ab}\mathbb{J}_{bc} \qquad [\mathbb{J}_{ab},\mathbb{J}_{bc}]=-\mathbb{J}_{ac} \qquad [\mathbb{J}_{ac},\mathbb{J}_{bc}]=\omega_{bc}\mathbb{J}_{ab} \tag{2.1}$$

where now $a<b<c, a,b,c=0,1,\dots,N$, $\omega_{ab}:=\omega_{a+1}\omega_{a+2}\dots\omega_b=\prod_{l=a+1}^{b}\omega_l$ (thus, $\omega_{ab}\omega_{bc}=\omega_{ac}$) and $[\mathbb{J}_{ab},\mathbb{J}_{cd}]=0$ if the four indices are different. By a simple rescaling of the generators, all the numerical values of the constants ω_i may be brought to one of the values $1,0,-1$, hence the complete CK family contains 3^N algebras which are different as graded contractions, even if some of them may still be isomorphic.

When all the ω_i are non-zero but some of them are negative, the algebra $so_{\omega_1,\omega_2,\dots,\omega_N}(N+1)$ is isomorphic to a certain pseudo-orthogonal algebra $so(p,q)$ $(p+q=N+1,\ p\geqslant q>0)$. If all the ω_i are non-zero we can also introduce $\mathbb{J}_{ba}$, $(a<b)$ by $\mathbb{J}_{ba}:=-\frac{1}{\omega_{ab}}\mathbb{J}_{ab}$ and $\mathbb{J}_{aa}:=0$, so that the commutation relations can be written in the familiar form:

$$[\mathbb{J}_{ij},\mathbb{J}_{lm}]=\delta_{im}\mathbb{J}_{lj}-\delta_{jl}\mathbb{J}_{im}+\delta_{jm}\omega_{lm}\mathbb{J}_{il}+\delta_{il}\omega_{ij}\mathbb{J}_{jm}. \tag{2.2}$$

If, however, some constant(s) $\omega_i=0$, the algebras (2.1) become inhomogeneous and correspond to algebras that are obtained from $so(p,q)$ through a sequence of IW

† We use the word 'affine' in the sense of inhomogeneous. Not all deformed inhomogeneous groups have a bicrossproduct structure; this is, for instance, the case of $\mathcal{U}_q(\mathcal{E}(2))$ as discussed in [20].

contractions. To describe them let us denote $\mathfrak{h}^{(m)}$ $(m = 1, \ldots, N)$ the subalgebra generated by the $\mathbb{J}_{ab}$ $(a < b)$ for which a, b satisfy either $b < m$ or $a \geqslant m$. A complement for $\mathfrak{h}^{(m)}$ is the vector subspace $\mathfrak{p}^{(m)}$ (not always a subalgebra) spanned by the elements $\mathbb{J}_{ab}$ with $a < m$ and $b \geqslant m$. The decomposition $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1) = \mathfrak{p}^{(m)} \oplus \mathfrak{h}^{(m)}$ is in fact a Cartan-like decomposition, and there exists an involutive automorphism of the Lie algebra $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ with $\mathfrak{p}^{(m)}$ and $\mathfrak{h}^{(m)}$ as the anti-invariant and invariant subspaces. The structure of the subalgebra $\mathfrak{h}^{(m)}$ and of the vector subspace $\mathfrak{p}^{(m)}$ of the Lie algebra $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ can be graphically displayed by arranging the generators of $so(N+1)$ in the form of a triangle

$$\begin{array}{cccc|cccc}
J_{01} & J_{02} & \ldots & J_{0(m-1)} & J_{0m} & J_{0(m+1)} & \ldots & J_{0N} \\
 & J_{12} & \ldots & J_{1(m-1)} & J_{1m} & J_{1(m+1)} & \ldots & J_{1N} \\
 & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & & & J_{(m-2)(m-1)} & J_{(m-2)m} & J_{(m-2)(m+1)} & \ldots & J_{(m-2)N} \\
 & & & & J_{(m-1)m} & J_{(m-1)(m+1)} & \ldots & J_{(m-1)N} \\
\cline{5-8}
\multicolumn{4}{c}{} & & J_{m(m+1)} & \ldots & J_{mN} \\
\multicolumn{4}{c}{} & & & \ddots & \vdots \\
\multicolumn{4}{c}{} & & & & J_{N-1N}.
\end{array} \tag{2.3}$$

We see that the generators which span the subspace $\mathfrak{p}^{(m)}$ are the $m(N+1-m)$ generators in the rectangle determined by the corner $J_{(m-1)m}$. The triangles at its left and below correspond to the subalgebras $so_{\omega_1,\ldots,\omega_{m-1}}(m)$ and $so_{\omega_{m+1},\ldots,\omega_N}(N+1-m)$ respectively, the direct sum of which is the subalgebra $\mathfrak{h}^{(m)}$. The subspace $\mathfrak{p}^{(m)}$ corresponding to the ω_m-rectangle in the diagram, can be identified with the Lie algebra quotient space $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)/so_{\omega_1,\ldots,\omega_{m-1}}(m) \oplus so_{\omega_{m+1},\ldots,\omega_N}(N+1-m)$.

For each decomposition $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1) = \mathfrak{p}^{(m)} \oplus \mathfrak{h}^{(m)}, m = 1, \ldots, N$, there is a possible IW contraction, denoted by $\Gamma^{(m)}$, to be performed on the algebra $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$. Specifically, if we denote the generators of the standard $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ algebra by $\mathbb{X}$, the IW contraction $\Gamma^{(m)}$ of $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ is given by the $\epsilon \to 0$ limit of the replacements

$$\Gamma^{(m)}(\mathbb{X}) \equiv \mathbb{X}' = \begin{cases} \mathbb{X} & \text{if } \mathbb{X} \in \mathfrak{h}^{(m)} \\ \epsilon\mathbb{X} & \text{if } \mathbb{X} \in \mathfrak{p}^{(m)} \end{cases} \qquad m = 1, \ldots, N \tag{2.4}$$

Under the contraction $\Gamma^{(m)}$, the algebra $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ goes to another algebra in the CK family with the same values of the ω_i constants except for $\omega_m = 0$. Thus, in the triangular arrangement of generators, the N possible IW contractions correspond to the N different rectangles that can be selected inside the large triangle. These rectangles are completely Abelianized by the contractions, while the commutators with one or two generators outside $\mathfrak{p}^{(m)}$ remain unchanged. As an example, the contraction given by (2.4) with $m = 1$ and starting from a $so(p, q)$ algebra, where all ω_i are different from zero, corresponds to the limit $\epsilon \to 0$ of $J_{0i} \mapsto J'_{0i} = \epsilon J_{0i}$, $J_{ij} = J'_{ij}(i \neq 0)$, $J_{0i}, J_{ij} \in so(p, q)$. This leads to $[J'_{0i}, J'_{0j}] = \pm\epsilon^2 J'_{ij}$ and hence J'_{0i} $i = 1, \ldots, N$ determines the Abelian N-dimensional ideal $\mathfrak{p}^{(1)}$.

Let us now consider the homogeneous space $\mathcal{S} \equiv SO_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)/SO_{\omega_2,\ldots,\omega_N}(N)$, where $SO_{\omega_2,\ldots,\omega_N}(N)$ is the subgroup generated by the subalgebra $\mathfrak{h}^{(m)}$ with $m = 1$. This space has an invariant canonical connection, and a hierarchy of metrics, coming after suitable rescalings from the Cartan–Killing form in the algebra $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$. When the constants $\omega_2, \ldots, \omega_N$ are different from zero, then the 'main' metric is non-degenerate, the invariant canonical connection turns out to be the corresponding Levi-Civita metric connection, and the space $\mathcal{S}$ has a curvature which is constant and equal to ω_1.

In the particular case $(\omega_1, \omega_2, \ldots, \omega_N) = (\omega_1, 1, \ldots, 1)$ the space $\mathcal{S}$ reduces to the Riemannian space (positive definite metric) of constant curvature ω_1 and dimension N. When $\omega_1 = 0$ the algebras $so_{0,\omega_2,\ldots,\omega_N}(N+1)$, can be realized as algebras of groups of affine transformations on $\mathbb{R}^N$ [12]; in this case we shall rename the generators as $\{\mathbb{P}_i := \mathbb{J}_{0i}, \mathbb{J}_{ij}; i < j, i, j = 1, \ldots, N\}$, the new names stressing the role of $\mathbb{P}_i$ as generating translations and of $\mathbb{J}_{ij}$ as the generators of rotations around the origin of the space. Each subalgebra $\mathfrak{h}^{(m)}, m = 1, \ldots, N$, is spanned by the set of generators $\{\mathbb{P}_i, \mathbb{J}_{ij}, i, j = 1, \ldots, m-1; \mathbb{J}_{kl}, k, l = m, \ldots, N\}$, and hence the collection of subalgebras $\mathfrak{h}^{(m)}$ can be clearly identified in the space $\mathcal{S}$ as the isotropy subalgebras of a point (for $m = 1$), of a line (for $m = 2$), ..., of a hyperplane (for $m = N$).

The non-zero Lie brackets of $so_{0,\omega_2,\ldots,\omega_N}(N+1)$ are given by

$$\begin{aligned} &[\mathbb{J}_{ij}, \mathbb{P}_i] = \mathbb{P}_j \qquad [\mathbb{J}_{ij}, \mathbb{P}_j] = -\omega_{ij}\mathbb{P}_i \qquad [\mathbb{J}_{ij}, \mathbb{J}_{ik}] = \omega_{ij}\mathbb{J}_{jk} \\ &[\mathbb{J}_{ij}, \mathbb{J}_{jk}] = -\mathbb{J}_{ik} \qquad [\mathbb{J}_{ik}, \mathbb{J}_{jk}] = \omega_{jk}\mathbb{J}_{ij} \end{aligned} \tag{2.5}$$

where the indices $i, j, k = 1, \ldots, N$ are always assumed to be ordered, $i < j < k$. Note in particular that all translation generators commute (as witnessing the zero curvature). It will be convenient to denote this $so_{0,\omega_2,\ldots,\omega_N}(N+1)$ Lie algebra by $iso_{\omega_2,\ldots,\omega_N}(N)$, and the corresponding group by $ISO_{\omega_2,\ldots,\omega_N}(N)$. There are 3^{N-1} different N-dimensional affine CK geometries, and from relations (2.5) it is clear that the groups $ISO_{\omega_2,\ldots,\omega_N}(N)$ have a semidirect product structure

$$ISO_{\omega_2,\ldots,\omega_N}(N) = SO_{\omega_2,\ldots,\omega_N}(N) \odot T_N \tag{2.6}$$

where T_N is the Abelian subgroup generated by $\{\mathbb{P}_i; i = 1, \ldots, N\}$ (in the case $\omega_1 = 0$, this Abelian subgroup can be identified with the CK homogeneous space $\mathcal{S}$ itself) and $SO_{\omega_2,\ldots,\omega_N}(N)$ is a general CK group with $N-1$ constants ω_i, generated by $\{\mathbb{J}_{ij}; i, j = 1, \ldots, N\}$. The 'main' metric which is kept invariant by the action of this group is described by the quadratic form given by a matrix with diagonal entries $(1, \omega_2, \omega_2\omega_3, \ldots, \omega_2 \ldots \omega_N)$. Among these inhomogeneous groups we can recognize the Euclidean group in N dimensions for which $(\omega_1, \omega_2, \ldots, \omega_N) = (0, 1, 1, \ldots, 1)$, the Poincaré group in $(N-1, 1)$ dimensions (appearing several times in the CK affine scheme as for example for $(0, -(1/c^2), 1, \ldots, 1))$ or the Galilei group in $(N-1, 1)$ dimensions which corresponds to the values $(\omega_1, \omega_2, \ldots, \omega_N) = (0, 0, 1, \ldots, 1)$; we recall that in all of these examples $\omega_1 = 0$. The geometrical meaning of the contractions $\Gamma^{(m)}$, $m = 1, \ldots, N$, is to describe the behaviour of the space $\mathcal{S} \equiv SO_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)/SO_{\omega_2,\ldots,\omega_N}(N)$ around a point, a line, ..., a hyperplane. In particular, within the inhomogeneous CK family $\omega_1 = 0$, only those contractions $\Gamma^{(m)}, m = 2, \ldots, N$ may produce a different algebra. In other words, these inhomogeneous algebras can be thought of as the result of a 'local' contraction (around a point, $m = 1$) which make the associated curvature vanish, although they can still be contracted to describe the behaviour of the space around a line, ..., a hyperplane, and hence the remaining contractions $\Gamma^{(m)}, m = 2, \ldots, N$ may be relevant. For instance, the non-relativistic limit, where the behaviour of spacetime geometry is approximated in the neighbourhood of a given (time-like) line corresponds to the contraction where $\omega_2 \to 0$.

A second-order central element for the algebra $iso_{\omega_2,\ldots,\omega_N}(N)$, coming after a specialization to this case of a suitable rescaling of the general CK Killing form, reads

$$\mathbb{C} = \sum_{i=1}^{N-1} \omega_{iN}\mathbb{P}_i^2 + \mathbb{P}_N^2. \tag{2.7}$$

Notice that this Casimir only involves generators from the Abelian translation subalgebra.

Summarizing, we see that the graded contraction language allows us to describe contractions simply by setting some parameters equal to zero. These contractions may still be described by the standard IW framework, although the graded contraction scheme is more economical and permits a unified discussion of the different contractions.

2.2. Contractions and dimensional analysis

The minimal possible approach to study the dimensional structure in CK algebras is done by enforcing the dimensional homogeneity of the commutation relations in all algebras in the CK family. In this approach, all generators as well as the structure constants are dimensional, in such a way that these dimensions are the same in all CK algebras. Consider the redefinition $\mathbb{J}_{ab} = \eta_{ab}\mathbb{J}'_{ab}$ for any CK algebra. If we now want the second commutator in (2.1) to be preserved (we still choose the structure constants equal to 1 as dimensionless, for we are interested here in algebras in the CK family, and not beyond), we need $\eta_{ab}\eta_{bc} = \eta_{ac}$, so that we see that η_{ab} may be expressed as $\eta_{ab} = \eta_{a+1}\eta_{a+2}\ldots\eta_b$. If we now make this change in the first and third commutators, we get

$$[\mathbb{J}'_{ab}, \mathbb{J}'_{ac}] = \frac{\omega_{ab}}{\eta^2_{ab}}\mathbb{J}'_{bc} \qquad [\mathbb{J}'_{ac}, \mathbb{J}'_{bc}] = \frac{\omega_{bc}}{\eta^2_{bc}}\mathbb{J}'_{ab}. \tag{2.8}$$

In the special case when all ω_i are different from zero (the case of simple algebras), the choice $\eta^2_{ab} = |\omega_{ab}|$ leads to the standard commutators for the $\mathbb{J}'_{ab}$ of the real form $so(p,q)$ of the specific algebra considered, with all non-zero structure constants equal to ± 1, and the $\mathbb{J}'_{ab}$'s are dimensionless.

In the general case (for generic CK algebras) by virtue of the above redefinition, the generators in (2.1) have as dimensions $[\mathbb{J}_{ab}] = [\omega_{ab}]^{1/2} = \prod_{i=a+1}^{b}[\omega_i]^{1/2}$. In this approach, each constant ω_a has dimensions, and if the dimension of the generator $\mathbb{J}_{a-1a}$ is written as D_a^{-1}, then it is clear that the dimensions of each ω_a are $[\omega_a] = D_a^{-2}$, irrespective of ω_a being zero or not. The dimension of each generator $\mathbb{J}_{ab}$ includes a factor $[\omega_a]$ for each of the ω_a-rectangles in (2.3) to which $\mathbb{J}_{ab}$ belongs.

Another possibility is to allocate dimensions to generators and/or canonical parameters for each CK algebra independently, in such a way as to make all non-zero structure constants in the algebra dimensionless. The idea of basing the dimensional analysis of a theory on the structure of its underlying Lie group/algebra has its roots in the well known examples of the Poincaré and Galilei groups, which are obtained by contracting with respect to two dimensionful parameters, the de Sitter radius and the velocity of light c, and has been discussed in [23]; see also [24].

If for a simple Lie algebra in the CK family (2.1) with non-zero ω_i constants we adopt this hypothesis, then as a consequence all the generators of the algebra, as well as their associated canonical parameters are also without dimensions (as the $\mathbb{J}'_{ab}$'s in the first approach). If the same requirement is applied to a non-simple CK Lie algebra, then we get the result that some generators are also dimensionless, while others get a dimension. For example, if this is done on a CK algebra with *a single* ω_a equal to zero, it is clear from the commutation relations that those generators which acquire in this case a non-trivial dimension D_a^{-1} (D_a is then the dimension of a corresponding canonical parameter) are exactly those inside the ω_a-rectangle corresponding to the constant ω_a which vanished in the triangular arrangement of generators.

If there are two constants equal to zero, say $\omega_a = \omega_b = 0$, there will be two non-trivial dimensions, and so on. Remark that now each ω_a has a dimension which is still D_a^{-1} when $\omega_a = 0$ but is dimensionless when $\omega_a \neq 0$. In this alternative choice the dimensions of the generators will still be given by $[\mathbb{J}_{ab}] = [\omega_{ab}]^{1/2} = \prod_{i=a+1}^{b}[\omega_i]^{1/2}$ but those factors where $\omega_i \neq 0$ are $[\omega_i] = 1$. This situation is exemplified in the transition from the Poincaré to the Galilei algebras. These are given by the values $(\omega_1, \omega_2, \omega_3, \omega_4) = (0, -1/c^2, 1, 1)$ and $(0, 0, 1, 1)$ respectively, and the arrangement of the Galilei and or Poincaré generators written in the usual physical notation is given by:

$$\begin{array}{cccc} H & P_1 & P_2 & P_3 \\ & K_1 & K_2 & K_3 \\ & & J_{12} & J_{13} \\ & & & J_{23} \end{array}. \qquad (2.9)$$

The assignment of dimensions made this way for Poincaré and for Galilei algebras is

$$\begin{array}{cccc} D_1^{-1} & D_1^{-1} & D_1^{-1} & D_1^{-1} \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{array} \qquad \begin{array}{cccc} D_1^{-1} & D_1^{-1}D_2^{-1} & D_1^{-1}D_2^{-1} & D_1^{-1}D_2^{-1} \\ & D_2^{-1} & D_2^{-1} & D_2^{-1} \\ & & 1 & 1 \\ & & & 1 \end{array}$$

which gives the single 'length' dimension in relativistic physics and the customary T, L dimensions of non-relativistic physics ($D_1 \equiv T$ and $D_2 \equiv LT^{-1}$, so $D_1 D_2 = L$).

The relation between both perspectives to the dimensional analysis of CK algebras is as follows. All constants ω_a can be considered at the beginning as dimensionful, and then all generators are also dimensionful. However, when a given $\omega_a \neq 0$, the dimension D_a can be removed by taking ω_a as a pure number, which can be set equal to ± 1; this is tantamount to fixing the scale of the generators or, in other words, to measuring the associated group parameter in terms of the corresponding unit much in the same way as in a relativistic theory we may adopt units in which $c = 1$ (i.e. $\omega_2 = -1$ above). In the former example, setting $\omega_3 = 1 = \omega_4$ may be understood as having hidden universal constants in the theory (cf [25]). However, once a dimensionful ω_a has been set equal to zero (i.e. a contraction has been made), the generators in the corresponding box retain a dimension $[\omega_a]^{1/2}$ since they cannot be rescaled any longer. This is why some generators in the former Galilei example retain the non-removable dimensions D_1, D_2, while D_2 disappears in the Poincaré case while D_3, D_4 have already disappeared in both cases†.

3. Deformed N-dimensional affine CK algebras

All the family of affine N-dimensional ($N \geqslant 2$) CK algebras $iso_{\omega_2,\ldots,\omega_N}(N)$ can be endowed with a standard deformed Hopf algebra structure which has been called a 'quantum' inhomogeneous CK structure and which has been given in [14, 15]. In order to avoid repeating statements on the index ranges, we will conform in sections 3 and 4 to the following convention: the range of a latin index i, j, k will be $1, \ldots, N-1$, and the index N will be dealt with separately, unless otherwise stated explicitly. Also, when two indices i, j appear in a generator, we will always assume that $i < j$.

Let $\mathcal{A}$ be the algebra of the formal power series in the deformation parameter λ with coefficients in the enveloping algebra $\mathcal{U}(iso_{\omega_2,\ldots,\omega_N}(N))$ of the Lie algebra $iso_{\omega_2,\ldots,\omega_N}(N)$ of (2.5). Then the coproduct, co-unit, antipode and deformed commutation relations of the algebra $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$, which is a Hopf algebra, are given by

† The above is not the only group-theoretical mechanism for the introduction of dimensions. Where centrally extended groups are physically relevant, the dimensions of the two-cocycle realizing the extension play a role. For instance, in the $(1+1)$-dimensional extended Galilei group we find two parameters characterizing the two-dimensional cohomology space, which correspond to the mass and a (constant) force.

(1) Coproduct:

$$\Delta(\mathbb{P}_N) = 1 \otimes \mathbb{P}_N + \mathbb{P}_N \otimes 1, \qquad \Delta(\mathbb{P}_i) = \mathrm{e}^{-\frac{\lambda}{2}\mathbb{P}_N} \otimes \mathbb{P}_i + \mathbb{P}_i \otimes \mathrm{e}^{\frac{\lambda}{2}\mathbb{P}_N}$$
$$\Delta(\mathbb{J}_{ij}) = 1 \otimes \mathbb{J}_{ij} + \mathbb{J}_{ij} \otimes 1$$
$$\Delta(\mathbb{J}_{iN}) = \mathrm{e}^{-\frac{\lambda}{2}\mathbb{P}_N} \otimes \mathbb{J}_{iN} + \mathbb{J}_{iN} \otimes \mathrm{e}^{\frac{\lambda}{2}\mathbb{P}_N} - \frac{\lambda}{2}\sum_{s=1}^{i-1} \mathbb{J}_{si}\mathrm{e}^{-\frac{\lambda}{2}\mathbb{P}_N} \otimes \omega_{iN}\mathbb{P}_s + \frac{\lambda}{2}\sum_{s=1}^{i-1} \omega_{iN}\mathbb{P}_s \otimes e^{\frac{\lambda}{2}\mathbb{P}_N}\mathbb{J}_{si}$$
$$+\frac{\lambda}{2}\sum_{s=i+1}^{N-1} \mathbb{J}_{is}\mathrm{e}^{-\frac{\lambda}{2}\mathbb{P}_N} \otimes \omega_{sN}\mathbb{P}_s - \frac{\lambda}{2}\sum_{s=i+1}^{N-1} \omega_{sN}\mathbb{P}_s \otimes \mathrm{e}^{\frac{\lambda}{2}\mathbb{P}_N}\mathbb{J}_{is}. \tag{3.1}$$

(2) Co-unit:

$$\varepsilon(\mathbb{P}_i) = \varepsilon(\mathbb{P}_N) = \varepsilon(\mathbb{J}_{ij}) = \varepsilon(\mathbb{J}_{iN}) = 0. \tag{3.2}$$

(3) Antipode:

$$\begin{aligned} &\gamma(\mathbb{P}_i) = -\mathbb{P}_i \qquad \gamma(\mathbb{P}_N) = -\mathbb{P}_N \\ &\gamma(\mathbb{J}_{ij}) = -\mathbb{J}_{ij} \qquad \gamma(\mathbb{J}_{iN}) = -\mathbb{J}_{iN} - \omega_{iN}(N-1)\frac{\lambda}{2}\mathbb{P}_i \end{aligned} \tag{3.3}$$

(it may be written in a compact way as $\gamma(\mathbb{X}) = -\mathrm{e}^{(N-1)\frac{\lambda}{2}\mathbb{P}_N}\mathbb{X}\mathrm{e}^{-(N-1)\frac{\lambda}{2}\mathbb{P}_N}$).

(4) Deformed commutators:

$$\begin{aligned} &[\mathbb{J}_{iN}, \mathbb{P}_j] = \delta_{ij}\frac{1}{\lambda}\sinh(\lambda\mathbb{P}_N) \\ &[\mathbb{J}_{iN}, \mathbb{J}_{jN}] = \omega_{jN}\left\{\mathbb{J}_{ij}\cosh(\lambda\mathbb{P}_N) + \frac{\lambda^2}{4}\left(\sum_{s=1}^{i-1}\omega_{iN}\mathbb{P}_s\mathbb{W}_{sij} - \sum_{s=i+1}^{j-1}\omega_{sN}\mathbb{P}_s\mathbb{W}_{isj}\right.\right. \\ &\qquad \left.\left. + \sum_{s=j+1}^{N-1}\omega_{sN}\mathbb{P}_s\mathbb{W}_{ijs}\right)\right\} \qquad i < j \end{aligned} \tag{3.4}$$

where

$$\mathbb{W}_{ijk} = \omega_{ij}\mathbb{P}_i\mathbb{J}_{jk} - \mathbb{P}_j\mathbb{J}_{ik} + \mathbb{P}_k\mathbb{J}_{ij} \qquad i < j < k \qquad i, j, k = 1, \ldots, N-1. \tag{3.5}$$

The remaining commutators are non-deformed and as given in (2.5). It may be checked that $(A, \Delta, \varepsilon, \gamma)$ satisfies the Hopf algebra axioms and hence equations (3.1)–(3.5) may be taken as the definition of the deformation $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ of $\mathcal{U}(iso_{\omega_2,\ldots,\omega_N}(N))$. The parameter λ has an inverse dimension to that of P_N so that the product $\lambda\mathbb{P}_N$ is dimensionless, and may be interpreted as the parameter left after contracting the deformed Hopf algebra $\mathcal{U}_q(so_{\omega_1,\ldots,\omega_N}(N+1))$ by previously redefining q in terms of λ and the contraction parameter. However, the expression of the deformation $\mathcal{U}_q(so_{\omega_1,\ldots,\omega_N}(N+1))$ in the 'physical' basis is not known and this precludes us for the moment from deriving (3.1)–(3.5) by contracting its deformed simple parent algebra $\mathcal{U}_q(so_{\omega_1,\ldots,\omega_N}(N+1))$. Nevertheless, it may be seen that the deformed Hopf algebra $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ is a quantization of the coboundary Lie bialgebra $(\mathcal{U}(iso_{\omega_2,\ldots,\omega_N}(N)), r)$ generated by the (non-degenerate) classical r-matrix

$$r = \lambda\sum_{s=1}^{N-1}\mathbb{J}_{sN} \wedge \mathbb{P}_s. \tag{3.6}$$

Due to the structure of r and our convention about dimensions, it turns out that r is dimensionless, regardless of the values of the constants ω_i since the product of J_{sN} and P_s will always have the same dimensions of P_N.

We remark that the above deformation (3.1)–(3.5) is not the only one possible for the $iso_{\omega_2,\ldots,\omega_N}(N)$ family. However, it is distinguished by the fact that all its members present deformed algebra and coalgebra sectors.

The quantum analogue of the second-order Casimir (2.7) is expressed by

$$\mathbb{C}_\lambda = \sum_{i=1}^{N-1} \omega_{iN}\mathbb{P}_i^2 + \frac{4}{\lambda^2}\left[\sinh\left(\frac{\lambda}{2}\mathbb{P}_N\right)\right]^2. \tag{3.7}$$

As far as their action on the algebra generators is concerned, the quantum versions $\Gamma_\lambda^{(m)}$ of the classical IW contractions $\Gamma^{(m)}$ are defined to coincide with the classical one (2.4). In particular, the generator $\mathbb{P}_N$ is rescaled by the corresponding contraction parameter ϵ in any of the contractions in the family $\Gamma_\lambda^{(m)}$, $\Gamma^{(m)}(\mathbb{P}_N) \equiv \mathbb{P}'_N = \epsilon\mathbb{P}_N$ (for $\epsilon \to 0$). This means that since one has to replace $\mathbb{P}_N$ in (3.1) by $\mathbb{P}'_N/\epsilon$, the exponents there will diverge. It is therefore natural to replace simultaneously λ by $\epsilon\lambda'$, i.e. to rescale the deformation parameter by $\Gamma^{(m)}(\lambda) \equiv \lambda' = \lambda/\epsilon$ (all primes are removed after taking the contraction limit), as the simplest possibility to preserve the coproduct (3.1). Therefore, the quantum contraction $\Gamma_\lambda^{(M)}$ is defined as the result of taking the limit $\epsilon \to 0$ in (3.1)–(3.4) once the transformations

$$\Gamma_\lambda^{(m)}(\epsilon, \mathbb{X}) = \Gamma^{(m)}(\epsilon, \mathbb{X}) \qquad \mathbb{X} = (\mathbb{P}, \mathbb{J}) \qquad \Gamma_\lambda^{(m)}(\epsilon, \lambda) = \lambda/\epsilon \tag{3.8}$$

are performed. We conclude this section with three observations. First, we have made a constant reference to the IW procedure only because up to now the graded contraction theory had not been extended to deformed algebras. Secondly, as far as the generators $\mathbb{X}$ are concerned, $\Gamma_\lambda^{(m)} = \Gamma^{(m)}$, so that only the action of $\Gamma_\lambda^{(m)}$ on λ makes $\Gamma^{(m)}$ and $\Gamma_\lambda^{(m)}$ different. The third comment is that the rescaling of λ and $\mathbb{P}_N$ implied by $\Gamma_\lambda^{(m)}$ may change their dimensions (see section 2) while consistently keeping a dimensionless $\lambda\mathbb{P}_N$ exponent.

4. Bicrossproduct structure of $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$

It is not obvious to see whether the Hopf algebra $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ has a bicrossproduct structure by a simple inspection of (3.1)–(3.4). The clue in this direction is provided by the bicrossproduct structure [21] of the κ-Poincaré algebra [5] (appearing in our scheme when $(\omega_1, \omega_2, \omega_3, \omega_4) = (0, 1, 1, -1)$), which is clearly displayed in terms of a new set of generators.

The aim of this section is to show that *all* the deformed Hopf algebras $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ in the CK family have indeed a bicrossproduct structure. The basic bicrossproduct formulae used are recalled in the appendix; for a detailed exposition, see [17]. Let P_i, J_{ij}, P_N, J_{iN} be the new set of generators defined in terms of the old ones $\mathbb{P}_i$ and $\mathbb{J}_{ij}$ by

$$\begin{aligned} &P_i = \mathrm{e}^{-(\lambda/2)\mathbb{P}_N}\mathbb{P}_i \qquad P_N = \mathbb{P}_N \qquad J_{ij} = \mathbb{J}_{ij} \\ &J_{iN} = \frac{1}{2}\{\mathbb{J}_{iN}, \mathrm{e}^{-(\lambda/2)\mathbb{P}_N}\} + \frac{\lambda}{4}\sum_{s=1}^{i-1}\omega_{iN}\{\mathbb{J}_{si}, \mathbb{P}_s\}\mathrm{e}^{-(\lambda/2)\mathbb{P}_N} - \frac{\lambda}{4}\sum_{s=i+1}^{N-1}\omega_{sN}\{\mathbb{J}_{is}, \mathbb{P}_s\}\mathrm{e}^{-(\lambda/2)\mathbb{P}_N} \end{aligned} \tag{4.1}$$

A straightforward but tedious computation leads to the following new expressions (where $i, j, k = 1, \ldots, N-1$) for the coproduct (3.1), co-unit (3.2), antipode (3.3) and algebra commutators ((3.4) and/or (2.5)):

(1) Coproduct:

$$\Delta(P_i) = \mathrm{e}^{-\lambda P_N} \otimes P_i + P_i \otimes 1 \qquad \Delta(P_N) = 1 \otimes P_N + P_N \otimes 1$$
$$\Delta(J_{ij}) = 1 \otimes J_{ij} + J_{ij} \otimes 1 \tag{4.2}$$
$$\Delta(J_{iN}) = \mathrm{e}^{-\lambda P_N} \otimes J_{iN} + J_{iN} \otimes 1 + \lambda \sum_{s=1}^{i-1} \omega_{iN} P_s \otimes J_{si} - \lambda \sum_{s=i+1}^{N-1} \omega_{sN} P_s \otimes J_{is}.$$

(2) Co-unit:

$$\varepsilon(P_i) = \varepsilon(P_N) = \varepsilon(J_{ij}) = \varepsilon(J_{iN}) = 0. \tag{4.3}$$

(3) Antipode:

$$\gamma(P_i) = -\exp(\lambda P_N) P_i \qquad \gamma(P_N) = -P_N \qquad \gamma(J_{ij}) = -J_{ij}$$
$$\gamma(J_{iN}) = -\mathrm{e}^{\lambda P_N} J_{iN} + \lambda \mathrm{e}^{\lambda P_N} \sum_{s=1}^{i-1} \omega_{iN} P_s J_{si} - \lambda \mathrm{e}^{\lambda P_N} \sum_{s=i+1}^{N-1} \omega_{sN} P_s J_{is}. \tag{4.4}$$

(4) Commutators:

$$[P_i, P_j] = 0 \qquad [P_i, P_N] = 0$$
$$[J_{ij}, J_{ik}] = \omega_{ij} J_{jk} \qquad [J_{ij}, J_{jk}] = -J_{ik} \qquad [J_{ik}, J_{jk}] = \omega_{jk} J_{ij}$$
$$[J_{ij}, J_{iN}] = \omega_{ij} J_{jN} \qquad [J_{ij}, J_{jN}] = -J_{iN} \qquad [J_{ik}, J_{jN}] = \omega_{jN} J_{ij}$$
$$[J_{ij}, P_k] = \delta_{ik} P_k - \delta_{jk} \omega_{ij} P_i \qquad [J_{ij}, P_N] = 0 \tag{4.5}$$
$$[J_{iN}, P_j] = \delta_{ij} \left(\frac{1 - \mathrm{e}^{-2\lambda P_N}}{2\lambda} - \frac{\lambda}{2} \sum_{s=1}^{N-1} \omega_{sN} P_s^2 \right) + \lambda \omega_{iN} P_i P_j \qquad [J_{iN}, P_N] = -\omega_{iN} P_i.$$

Thus, all brackets for the new generators P_i, P_N, J_{ij}, J_{iN} coincide with the non-deformed ones given in (2.5) (substituting everywhere the new X's for their counterparts $\mathbb{X}$) except for $[J_{iN}, P_j]$, which is now the only deformed commutation relation. The effect of (4.1) is to modify the second commutator in (3.4), so that one recovers the undeformed $so_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ algebra commutators, and to replace the commutators in the first line of (3.4) by those in the last line of (4.5). As a result, terms with the $\mathbb{W}$ symbols are no longer present in the deformed commutators.

It may be checked that for $\lambda = 1/\kappa$ and $N = 4$ with $(\omega_1, \omega_2, \omega_3, \omega_4) = (0, 1, 1, -1)$ equations (4.2)–(4.5) reproduce the κ-Poincaré algebra in the basis of [21] for which $[\kappa] = L^{-1}$, $[P_4] = L^{-1}$. If we want P_4 to have dimensions of inverse time we may take $(\omega_1, \omega_2, \omega_3, \omega_4) = (0, 1, 1, -c^2)$ instead since ω_1, before being set equal to zero, was $\omega_1 = 1/R^2$; in this case $[\kappa] = T^{-1}$. We check that the metric after (2.6) will diverge in a non-relativistic limit with $\omega_4 = -c^2$, which explains why a non-relativistic limit of the κ-Poincaré algebra [5] requires a further redefinition of the deformation parameter κ (see the end of section 5).

The new expressions for the coproduct, co-unit, antipode and commutation relations of $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ now allow us to uncover its bicrossproduct structure. For this aim, consider the translation sector, generated by $\{P_1, \ldots, P_N\}$. According to expressions (4.2)–(4.5), it defines a commutative but non-cocommutative Hopf subalgebra of $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ which will be denoted as $\mathcal{U}_\lambda(T_N)$. Now let $\mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N))$ be the non-commutative and cocommutative non-deformed CK Hopf algebra spanned by the remaining generators $\{J_{ij}; i < j, i, j = 1, \ldots, N\}$, hence with commutation relations given by (2.5) and primitive coproduct (when all ω's are non-zero, this is a pseudo-orthogonal algebra). Let us define a right action $\alpha : \mathcal{U}_\lambda(T_N) \otimes \mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N)) \to \mathcal{U}_\lambda(T_N)$ by

$$\alpha(P_i, J_{jk}) \equiv P_i \triangleleft J_{jk} := [P_i, J_{jk}] \qquad j < k,\ i, j, k = 1, 2, \ldots, N \tag{4.6}$$

where the commutators are given in (4.5), and a left coaction $\beta : \mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N)) \to \mathcal{U}_\lambda(T_N) \otimes \mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N))$ by

$$\begin{aligned} \beta(J_{ij}) &:= 1 \otimes J_{ij} \\ \beta(J_{iN}) &:= \mathrm{e}^{-\lambda P_N} \otimes J_{iN} + \lambda \sum_{s=1}^{i-1} \omega_{iN} P_s \otimes J_{si} - \lambda \sum_{s=i+1}^{N-1} \omega_{sN} P_s \otimes J_{is}. \end{aligned} \tag{4.7}$$

It may be checked that $\mathcal{U}_\lambda(T_N)$ is a right $\mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N))$-module algebra $(\mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N)) \triangleright\!< \mathcal{U}_\lambda(T_N))$ and that $\mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N))$ is a left $\mathcal{U}_\lambda(T_N)$-comodule coalgebra $(\mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N)) >\!\!\blacktriangleleft \mathcal{U}_\lambda(T_N))$ under the action (4.6) and coaction (4.7), respectively, and that the compatibility conditions [17] (A.1)–(A.5) between α y β needed for $\mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N)) \otimes \mathcal{U}_\lambda(T_N)$ to have a bicrossproduct structure are fulfilled. For instance, (A.5) is automatically satisfied, since $\mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N))$ is undeformed and hence cocommutative and $\mathcal{U}_\lambda(T_N)$ is Abelian. This case, especially relevant here, was discussed in [18]. Then, the bicrossproduct structure of $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ may be stated in the form of the following.

Theorem. The deformed Hopf CK family of algebras $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ has a bicrossproduct structure

$$\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N)) = \mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N))^\beta \triangleright\!\!\blacktriangleleft_\alpha \mathcal{U}_\lambda(T_N)$$

relative to the right action α and left coaction β given by (4.6) and (4.7) respectively.

Proof. As mentioned, the mappings α and β satisfy the bicrossproduct conditions as may be checked by direct computation. Then expressions (A.6)–(A.9) give the associated coproduct, co-unit and antipode. It is then verified that the resulting expressions are in agreement with (4.2)–(4.5). □

The interesting consequence of the above discussion is that, as the direct inspection of expressions (4.6) and (4.7) shows, the action and the coaction mappings depend on the parameters ω_i in such a way that the bicrossproduct structure is formally invariant under any contraction $\omega_i = 0$. In other words, for λ-deformations in the affine CK family, the bicrossproduct structure is preserved by *all* the successive contractions: contracting and taking bicrossproduct of the appropriate Hopf algebras with the resulting actions and coactions are commuting processes. This is well within the spirit of the CK scheme, the aim of which is to state properties which hold simultaneously for a large number of algebras.

The expression of the deformed Casimir (3.7) in the new basis is

$$\mathcal{C}_\lambda = \sum_{i=1}^{N-1} \omega_{iN} \mathrm{e}^{-\lambda P_N} P_i^2 + \frac{4}{\lambda^2} \left[\sinh\left(\frac{\lambda}{2} P_N\right)\right]^2 \tag{4.8}$$

it only depends on the generators of the deformed Hopf subalgebra $\mathcal{U}_\lambda(\mathbb{T}_N)$.

On the other hand, the expression for the r-matrix is similar to the former (3.6) but in terms of the new generators,

$$r = \lambda \sum_{s=1}^{N-1} J_{sN} \wedge P_s. \tag{4.9}$$

5. Applications

The quantum algebras we are dealing with range from deformations of the inhomogeneous algebras $iso(p, q)$, $p + q = N$ (when all constants $\omega_2, \omega_3, \ldots \omega_N$ are different from zero) to the extreme case of a Hopf deformation of the algebra, where all constants are equal to zero, which can be called flag space algebra (in this case the group action preserves a complete flag).

Classically, all these algebras are semidirect products, and indeed there is a semidirect structure in the CK algebras associated to the vanishing of each constant ω_i. We have restricted ourselves here to the algebras with $\omega_1 = 0$, all of which have the semidirect structure displayed in (2.5). When *all* remaining constants ω_i are different from zero, say $\omega_i = \pm 1$, the algebra $iso_{\omega_2,\ldots,\omega_N}(N)$ is isomorphic to an inhomogeneous pseudo-orthogonal algebra $iso(p,q)$, $p+q=N$, with the semidirect structure given by the natural action of $so(p,q)$ on $\mathbb{R}^N$. These algebras are physically very relevant and some of their deformations have been thoroughly studied. In particular, the λ-deformed structures given in (3.1)–(3.4) include a deformed N-dimensional Euclidean algebra, a deformed $(N-1,1)$ Galilei algebra and several deformed $(N-1,1)$ Poincaré algebras, as well as their analogues for any signature.

The action and coaction mappings associated with the bicrossproduct are given by (4.6) and (4.7). Explicitly, equation (4.5) gives

$$\begin{aligned}
&\alpha(P_N, J_{ij}) \equiv P_N \triangleleft J_{ij} := 0 \qquad \alpha(P_N, J_{iN}) \equiv P_N \triangleleft J_{iN} := \omega_{iN} P_i \\
&\alpha(P_k, J_{ij}) \equiv P_k \triangleleft J_{ij} := -\delta_{ki} P_j + \delta_{kj}\omega_{ij} P_i \\
&\alpha(P_k, J_{iN}) \equiv P_k \triangleleft J_{iN} := -\delta_{ki}\left(\frac{1-\mathrm{e}^{-2\lambda P_N}}{2\lambda} - \frac{\lambda}{2}\sum_{s=1}^{N-1}\omega_{sN} P_s^2\right) - \lambda\omega_{iN} P_i P_k.
\end{aligned} \tag{5.1}$$

If we consider the special case where $N = 4$, this set of algebras includes *four* deformed Poincaré algebra $\mathcal{U}_\lambda(\mathfrak{p}^{(s)}(3,1))$, $s = 1,2,3,4$. These are deformations of the four undeformed CK algebras, denoted as $\mathfrak{p}^{(s)}(3,1)$, $s = 1,2,3,4$, which are isomorphic to the $(3,1)$ Poincaré algebra, and correspond to identifying one of the generators P_i to the time translation generator, the other three being space translations. If the time generator is taken successively to be our P_1, P_2, P_3, P_4, these four algebras correspond to the four sets of values of $(\omega_1,\omega_2,\omega_3,\omega_4) = (0,-1/c^2,1,1)$, $(0,-c^2,-1/c^2,1)$, $(0,1,-c^2,-1/c^2)$, and $(0,1,1,-c^2)$. The set of four deformed Poincaré algebras $\mathcal{U}_\lambda(\mathfrak{p}^{(s)}(3,1))$, $s = 1,2,3,4$ [15] includes three 'space-like' Poincaré deformed algebras, the last one being the κ-Poincaré algebra once $\lambda = 1/\kappa$ with $[P_N] = T^{-1}$). In each case, the rotation generators comprise the boost and space rotation generators and the identification is made according to the choice of the time generator (e.g. in the κ-Poincaré the boosts are the $N_i = J_{i4}$). The N-dimensional κ-Poincaré [26] is associated to the ω_i values $(0,1,\ldots,1,-c^2)$.

The Euclidean algebra $\mathfrak{e}(4)$ appears only once (up to rescalings) for $(\omega_1,\omega_2,\omega_3,\omega_4) = (0,1,1,1)$ and the bicrossproduct structure of their Hopf CK quantum deformation $\mathcal{U}_\lambda(\mathfrak{e}(4))$, is

$$\mathcal{U}_\lambda(\mathfrak{e}(4)) = \mathcal{U}(so(4)) \triangleright\!\!\blacktriangleleft \mathcal{U}_\lambda(\mathbb{R}_4)$$

(see [20] in the lower-dimensional case).

The remaining quantum Hopf algebras in the CK family are quantum deformations of less known undeformed Lie algebras, yet their bicrossproduct structure is described in parallel to the former cases. Especially relevant from the physical point of view is the Galilei algebra $\mathfrak{g}(3,1)$, appearing within the affine CK family for the ω values $(0,0,1,1)$ (and only for these). It is worth remarking that this Galilei algebra is obtained from the $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3,1))$ associated to the ω values $(0,-1/c^2,1,1)$ by means of the contraction $\omega_2 \to 0$ (i.e. $c \to \infty$), which gives a deformation different from that in [27, 28]. The bicrossproduct structure of the resulting $\mathcal{U}_\lambda(\mathfrak{g}(3,1))$ Hopf algebra (obtained for $(0,0,1,1)$ is

$$\mathcal{U}_\lambda(\mathfrak{g}(3,1)) = \mathcal{U}(so_{(0,1,1)}(4)) \triangleright\!\!\blacktriangleleft \mathcal{U}_\lambda(\mathbb{R}_4) \equiv \mathcal{U}(iso_{(1,1)}(3)) \triangleright\!\!\blacktriangleleft \mathcal{U}_\lambda(\mathbb{R}_4).$$

It is interesting to check how the action and coaction mappings for the Poincaré Hopf algebra $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3,1))$ reduce to the corresponding Galilean ones under the contraction $\omega_2 \to 0$: $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3,1))$ to $\mathcal{U}_\lambda(\mathfrak{g}(3,1))$. We give explicitly the complete Hopf structure of both $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3,1))$ and $\mathcal{U}_\lambda(\mathfrak{g}(3,1))$, which correspond to the choices $(0,-1/c^2,1,1)$ and $(0,0,1,1)$ for the ω_i's. We will present the results in the usual physical basis, constituted by the generators of time translation, H, space translations P_1, P_2, P_3, boosts K_1, K_2, K_3, and space rotations J_1, J_2, J_3, which are related to the CK original generators $\{P_i, J_{ij}; i, j = 1,2,3,4\}$ as follows. The three *space* translations, now denoted as P_1, P_2, P_3 in order to conform with the standard physical notation, correspond to those formerly denoted as P_2, P_3, P_4 in (4.2)–(4.5), while the time translation generator H now corresponds to the former P_1 and the rest in (4.2)–(4.5) correspond to $J_{12} = K_1, J_{13} = K_2, J_{14} = K_3, J_{34} = J_1, J_{24} = -J_2, J_{23} = J_3$, as given in diagram (2.9).

The Hopf structure of the Poincaré algebra $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3,1))$ now follows from expressions (4.2)–(4.5):

(1) Coproduct:

$$\begin{aligned}
&\Delta(X) = \mathrm{e}^{-\lambda P_3} \otimes X + X \otimes 1 \qquad X \in \{H, P_1, P_2\}\\
&\Delta(P_3) = 1 \otimes P_3 + P_3 \otimes 1\\
&\Delta(X) = 1 \otimes X + X \otimes 1 \qquad X \in \{K_1, K_2, J_3\}\\
&\Delta(K_3) = \mathrm{e}^{-\lambda P_3} \otimes K_3 + K_3 \otimes 1 - \lambda P_1 \otimes K_1 - \lambda P_2 \otimes K_2\\
&\Delta(J_1) = \mathrm{e}^{-\lambda P_3} \otimes J_1 + J_1 \otimes 1 + \lambda H \otimes K_2 + \lambda P_1 \otimes J_3\\
&\Delta(J_2) = \mathrm{e}^{-\lambda P_3} \otimes J_2 + J_2 \otimes 1 - \lambda H \otimes K_1 + \lambda P_2 \otimes J_3.
\end{aligned} \tag{5.2}$$

(2) Co-unit:

$$\varepsilon(H) = \varepsilon(P_i) = \varepsilon(K_i) = \varepsilon(J_i) = 0 \qquad i = 1,2,3. \tag{5.3}$$

(3) Antipode:

$$\begin{aligned}
&\gamma(X) = -\mathrm{e}^{\lambda P_3} X \qquad X \in \{H, P_1, P_2\} \qquad \gamma(P_3) = -P_3\\
&\gamma(X) = -X \qquad X \in \{K_1, K_2, J_3\}\\
&\gamma(K_3) = -\mathrm{e}^{\lambda P_3} K_3 - \lambda \mathrm{e}^{\lambda P_3} P_1 K_1 - \lambda \mathrm{e}^{\lambda P_3} P_2 K_2\\
&\gamma(J_1) = -\mathrm{e}^{\lambda P_3} J_1 + \lambda \mathrm{e}^{\lambda P_3} H K_2 + \lambda \mathrm{e}^{\lambda P_3} P_1 J_3\\
&\gamma(J_2) = -\mathrm{e}^{\lambda P_3} J_2 - \lambda \mathrm{e}^{\lambda P_3} H K_1 + \lambda \mathrm{e}^{\lambda P_3} P_2 J_3.
\end{aligned} \tag{5.4}$$

(4) Commutators:

$$
\begin{aligned}
&[H, P_i] = 0 \qquad [P_i, P_j] = 0 \qquad i, j = 0, 1, 2, 3\\
&[K_i, H] = P_i \qquad i = 1, 2 \qquad [K_3, H] = \frac{1 - \mathrm{e}^{-2\lambda P_3}}{2\lambda} - \frac{\lambda}{2c^2} H^2 - \frac{\lambda}{2} P_1^2 - \frac{\lambda}{2} P_2^2\\
&[K_i, P_j] = \frac{1}{c^2}\delta_{ij} H \qquad i, j = 1, 2 \qquad [K_i, P_3] = 0\\
&[K_3, P_1] = -\frac{\lambda}{c^2} H P_1 \qquad [K_3, P_2] = -\frac{\lambda}{c^2} H P_2 \qquad [K_3, P_3] = \frac{1}{c^2} H\\
&[J_1, H] = \lambda P_2 H \qquad [J_2, H] = -\lambda P_1 H \qquad [J_3, H] = 0\\
&[J_1, P_1] = \lambda P_2 P_1 \qquad [J_1, P_2] = \frac{1 - \mathrm{e}^{-2\lambda P_3}}{2\lambda} + \frac{\lambda}{2c^2} H^2 - \frac{\lambda}{2} P_1^2 + \frac{\lambda}{2} P_2^2\\
&\qquad\qquad [J_1, P_3] = -P_2\\
&[J_2, P_1] = -\frac{1 - \mathrm{e}^{-2\lambda P_3}}{2\lambda} - \frac{\lambda}{2c^2} H^2 - \frac{\lambda}{2} P_1^2 + \frac{\lambda}{2} P_2^2 \qquad [J_2, P_2] = -\lambda P_1 P_2\\
&\qquad\qquad [J_2, P_3] = P_1\\
&[J_3, P_1] = P_2 \qquad [J_3, P_2] = -P_1 \qquad [J_3, P_3] = 0\\
&[K_i, K_j] = -\frac{1}{c^2}\varepsilon_{ijk} K_k \qquad [J_i, K_j] = \varepsilon_{ijk} K_k \qquad [J_i, J_j] = \varepsilon_{ijk} J_k \qquad i, j = 1, 2, 3.
\end{aligned}
\tag{5.5}
$$

The explicit form of the action and coaction for $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3, 1))$, given by (4.6) and (4.7) is obtained from the commutators (5.5) and the coproduct (5.2). The bicrossproduct structure of $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3, 1))$ is

$$
\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3, 1)) = \mathcal{U}(so_{(-1/c^2,1,1)}(4)) \bowtie\!\!\blacktriangleleft \mathcal{U}_\lambda(\mathbb{R}_4).
$$

In the non-relativistic limit, $\omega_2 = -1/c^2 \to 0$, this deformed algebra goes to a new deformed Galilei algebra, $\mathcal{U}_\lambda(\mathfrak{g}(3, 1))$, whose coproduct, co-unit and antipode are the same as in (5.2)–(5.4). With respect to the Lie commutators we write only those that are different from the Poincaré $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3, 1))$ ones in (5.5)

$$
\begin{aligned}
&[K_3, H] = \frac{1 - \mathrm{e}^{-2\lambda P_3}}{2\lambda} - \frac{\lambda}{2} P_1^2 - \frac{\lambda}{2} P_2^2\\
&[K_i, P_j] = 0 \qquad i, j = 1, 2, 3\\
&[J_1, P_2] = \frac{1 - \mathrm{e}^{-2\lambda P_3}}{2\lambda} - \frac{\lambda}{2} P_1^2 + \frac{\lambda}{2} P_2^2\\
&[J_2, P_1] = -\frac{1 - \mathrm{e}^{-2\lambda P_3}}{2\lambda} - \frac{\lambda}{2} P_1^2 + \frac{\lambda}{2} P_2^2\\
&[K_i, K_j] = 0 \qquad i, j = 1, 2, 3.
\end{aligned}
\tag{5.6}
$$

From simple inspection we see that the Galilean action (as the commutators) has changed from that in $\mathcal{U}_\lambda(\mathfrak{p}^{(1)}(3, 1))$, yet the coaction is the same as in Poincaré since the coproduct has remained the same.

All other Poincaré quantum algebras (including the κ-Poincaré) do not allow a direct ‘non-relativistic’ contraction; this is clear when the constant c is explicitly written as above (see [28] for a discussion).

6. The group deformation aspect: the case of $\mathbf{Fun}_\lambda(ISO_{\omega_2}(2))$

The bicrossproduct structure of $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ opens the way to the possibility of recovering the dual group $\mathrm{Fun}_\lambda(ISO_{\omega_1,\omega_2,\ldots,\omega_N}(N))$ more easily from the dual bicrossproduct 'group-like' expressions. We show here the explicit calculation in the lowest dimension $N = 2$ to exhibit the procedure. As shown in section 4, the deformed Hopf algebra has the bicrossproduct structure

$$\mathcal{U}_\lambda(iso_{\omega_2}(2)) = \mathcal{U}(so_{\omega_2}(2)) \bowtie\!\!\!\blacktriangleleft \mathcal{U}_\lambda(\mathbb{T}_2). \tag{6.1}$$

Let us now recover from it the quantum dual group, $\mathrm{Fun}_\lambda(ISO_{\omega_2}(2))$.

The dual algebra $\mathrm{Fun}_\lambda(\mathbb{T}_2)$ of $\mathcal{U}_\lambda(\mathbb{T}_2)$ is easily found from $\mathcal{U}_\lambda(\mathbb{T}_2)$ to be

$$\Delta(a_1) = 1 \otimes a_1 + a_1 \otimes 1 \qquad \Delta(a_2) = 1 \otimes a_2 + a_2 \otimes 1 \tag{6.2}$$

$$\varepsilon(a_i) = 0 \qquad \gamma(a_i) = -a_i \qquad i = 1, 2 \tag{6.3}$$

and

$$[a_1, a_2] = \lambda a_1 \tag{6.4}$$

where a_1, a_2 are a system of coordinates of $\mathrm{Fun}_\lambda(\mathbb{T}_2)$.

On the other hand, let $\mathrm{Fun}(SO_{\omega_2}(2))$ be the dual of $\mathcal{U}(SO_{\omega_2}(2))$ generated by φ, with a non-deformed Hopf structure defined by

$$\Delta(\varphi) = 1 \otimes \varphi + \varphi \otimes 1 \qquad \varepsilon(\varphi) = 0 \qquad \gamma(\varphi) = -\varphi. \tag{6.5}$$

Now, the problem is finding a pair of mappings $\bar{\beta}$ and $\bar{\alpha}$ (see the appendix),

$$\begin{aligned} \bar{\beta} &: \mathrm{Fun}_\lambda(\mathbb{T}_2) \to \mathrm{Fun}_\lambda(\mathbb{T}_2) \otimes \mathrm{Fun}(SO_{\omega_2}(2)) \\ \bar{\alpha} &: \mathrm{Fun}_\lambda(\mathbb{T}_2) \otimes \mathrm{Fun}(SO_{\omega_2}(2)) \to \mathrm{Fun}(SO_{\omega_2}(2)) \end{aligned} \tag{6.6}$$

duals, respectively, to α and β as given in (5.1), (4.7) for $N = 2$. A calculation shows that they have the form

$$\begin{aligned} \bar{\beta}(a_1) &= a_1 \otimes C_{\omega_2}(\varphi) + a_2 \otimes \omega_2 S_{\omega_2}(\varphi) \\ \bar{\beta}(a_2) &= -a_1 \otimes S_{\omega_2}(\varphi) + a_2 \otimes C_{\omega_2}(\varphi) \end{aligned} \tag{6.7}$$

and

$$\bar{\alpha}(a_1 \otimes \varphi) = \lambda(1 - C_{\omega_2}(\varphi)) \qquad \bar{\alpha}(a_2 \otimes \varphi) = \lambda S_{\omega_2}(\varphi) \tag{6.8}$$

where the functions $C_\omega(\varphi)$ and $S_\omega(\varphi)$ reduce to the trigonometric cosine and sine functions for $\omega = 1$ and to the hyperbolic ones for $\omega = -1$ (see [29] for more details). Note that

$$(\bar{\beta}(a_1), \bar{\beta}(a_2)) = (a_1, a_2)\dot{\otimes} \begin{pmatrix} C_{\omega_2}(\varphi) & -S_{\omega_2}(\varphi) \\ \omega_2 S_{\omega_2}(\varphi) & C_{\omega_2}(\varphi) \end{pmatrix}$$

where the 2×2 matrix is the transpose of the matrix representing the generic element of the group $SO_{\omega_2}(2)$.

Since $\bar{\beta}$ modifies the originally cocommutative coproduct in $\mathrm{Fun}_\lambda(\mathbb{T}_2)$ and $\bar{\alpha}$ the commutation relations between the generators of the two algebras $\mathrm{Fun}_\lambda(\mathbb{T}_2)$ and that generated by φ (see equations (A$'$.7) and (A$'$.6)), we may now determine through its bicrossproduct structure the deformed Hopf algebra $\mathrm{Fun}_\lambda(SO_{\omega_2}(2+1))$ which has the form

$$\begin{aligned} \Delta(a_1) &= 1 \otimes a_1 + a_1 \otimes C_{\omega_2}(\varphi) + a_2 \otimes \omega_2 S_{\omega_2}(\varphi) \\ \Delta(a_2) &= 1 \otimes a_2 - a_1 \otimes S_{\omega_2}(\varphi) + a_2 \otimes C_{\omega_2}(\varphi) \\ \Delta(\varphi) &= 1 \otimes \varphi + \varphi \otimes 1 \end{aligned} \tag{6.9}$$

$$\varepsilon(a_i) = 0 \qquad i = 1, 2 \qquad \varepsilon(\varphi) = 0 \tag{6.10}$$

$$\begin{aligned} \gamma(a_1) &= -C_{\omega_2}(\varphi)a_1 - \omega_2 S_{\omega_2}(\varphi)a_2 \\ \gamma(a_2) &= S_{\omega_2}(\varphi)a_1 - C_{\omega_2}(\varphi)a_2 \end{aligned} \tag{6.11}$$

$$[a_1, \varphi] = \lambda(1 - C_{\omega_2}(\varphi)) \qquad [a_2, \varphi] = \lambda S_{\omega_2}(\varphi) \qquad [a_1, a_2] = \lambda a_1. \tag{6.12}$$

In this way the results obtained in [29] (and [20] for $\omega_2 = 1, a_1 \to -a_2, a_2 \to a_1$) are recovered.

7. Conclusions

The bicrossproduct structure is true for all deformed Hopf algebras in the inhomogeneous CK family $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$. The theorem in section 4 follows from the fact that the action and the coaction mappings that characterize the bicrossproduct structure of $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ depend on $\omega_2, \omega_3, \ldots, \omega_N$ in such a way that the contractions are simply described by setting (some of) them equal to zero. In this sense we may say that the bicrossproduct structure (4.6), (4.7) is compatible with the contractions (3.8). As a result, any contracted λ-algebra is also a bicrossproduct for the 'contracted' action and coaction. We can also state this result by saying that the bicrossproduct structure does not diverge under any of the contractions within the CK affine family. These results extend obviously to the dual $\mathrm{Fun}_\lambda(ISO_{\omega_2,\ldots,\omega_N}(N))$ family of deformed Hopf algebras.

The deformation parameter λ behaves in the same way under the complete family of CK contractions of the inhomogeneous algebras $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$; if the (first, for instance) contraction has been realized as a limit, the deformation parameter z in $q = \mathrm{e}^z$ is redefined by $\Gamma_\lambda^{(m)}(z) = z' = \frac{\lambda}{R}$ (cf equation (3.8)) and the limit corresponds to $[\omega_1]^{1/2} = 1/R \to 0$. As discussed, this can be understood as a mechanism assigning dimensions to the deformation parameter characterizing the deformed $\mathcal{U}_\lambda(iso_{\omega_2,\ldots,\omega_N}(N))$ CK family. In any case, we wish to stress here that the dimensions of λ which result from the *first* contraction process (that involving ω_1) depend crucially on the assumed dependence of the dimensionless q (or z) on the new deformation parameter λ and of the contraction one. For instance, if $q = \exp(\lambda/R)$ as above, where R is a radius (length), $[\lambda] = L^1$. Any other dimensions assigned to λ necessarily include hidden hypotheses on the dependence of q on *other* fundamental constants ($\hbar$ and c are necessary, for instance, in order to have $[\lambda]$= (mass)$^{-1}$). This is especially important because the appearance of Planck's $\hbar$, for instance, implies quantum considerations in the strict (i.e. physical) sense of the word. These go beyond the purely mathematical deformation process, and should be accordingly discussed separately in any physical application of a λ-deformed algebra (see [30] in connection with deformed Minkowski spaces).

It would be interesting to know whether the bicrossproduct structure is present in other cases, i.e. for $\omega_1 \neq 0$ different from the $\omega_1 = 0$ deformations considered here. However, the general form of the deformation $\mathcal{U}_q(so_{\omega_1,\ldots,\omega_N}(N+1))$ of the general CK algebra in the 'physical' (rather than the Cartan–Weyl) basis is unknown, and this precludes us for the moment from discussing this point.

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Appendix: Some bicrossproduct formulae

We reproduce here some of the formulae needed in the main text, and refer to [17] for details. In these formulae $\mathcal{A} \equiv \mathcal{U}_\lambda(T_N)$ and $\mathcal{H} \equiv \mathcal{U}(so_{\omega_2,\ldots,\omega_N}(N))$. The right action is $\alpha : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A}, \alpha(a \otimes h) \equiv a \triangleleft h$, and the left coaction $\beta : \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}, \beta(h) = h^{(1)} \otimes h^{(2)}, (h^{(1)} \in \mathcal{A}, h^{(2)} \in \mathcal{H})$ so that $(\mathcal{A}, \alpha)$ $[(\mathcal{H}, \beta)]$ is a right $\mathcal{H}$-module [left $\mathcal{A}$-comodule]. The compatibility conditions are

$$\varepsilon_{\mathcal{A}}(a \triangleleft h) = \varepsilon_{\mathcal{A}}(a)\varepsilon_{\mathcal{H}}(h) \tag{A.1}$$

$$\Delta(a \triangleleft h) \equiv (a \triangleleft h)_{(1)} \otimes (a \triangleleft h)_{(2)} = (a_{(1)} \triangleleft h_{(1)})h_{(2)}^{(1)} \otimes a_{(2)} \triangleleft h_{(2)}^{(2)} \tag{A.2}$$

$$\beta(1_{\mathcal{H}}) \equiv 1_{\mathcal{H}}^{(1)} \otimes 1_{\mathcal{H}}^{(2)} = 1_{\mathcal{A}} \otimes 1_{\mathcal{H}} \tag{A.3}$$

$$\beta(hg) \equiv (hg)^{(1)} \otimes (hg)^{(2)} = (h^{(1)} \triangleleft g_{(1)})g_{(2)}^{(1)} \otimes h^{(2)}g_{(2)}^{(2)} \tag{A.4}$$

$$h_{(1)}^{(1)}(a \triangleleft h_{(2)}) \otimes h_{(1)}^{(2)} = (a \triangleleft h_{(1)})h_{(2)}^{(1)} \otimes h_{(2)}^{(2)} \tag{A.5}$$

(subindices refer to coproducts as usual; *superindices* refer to the components of $\beta(h)$). When they are satisfied the right–left bicrossproduct structure $\mathcal{H}^{\beta} \triangleright\!\!\blacktriangleleft_{\alpha} \mathcal{A}$ on $\mathcal{K} \equiv \mathcal{H} \otimes \mathcal{A}$ is determined by

$$(h \otimes a)(g \otimes b) = hg_{(1)} \otimes (a \triangleleft g_{(2)})b \qquad h, g \in \mathcal{H}, \ a, b \in \mathcal{A} \tag{A.6}$$

$$\Delta_{\mathcal{K}}(h \otimes a) = h_{(1)} \otimes h_{(2)}^{(1)}a_{(1)} \otimes h_{(2)}^{(2)} \otimes a_{(2)} \tag{A.7}$$

$$\varepsilon_{\mathcal{K}} = \varepsilon_{\mathcal{H}} \otimes \varepsilon_{\mathcal{A}} \qquad 1_{\mathcal{K}} = 1_{\mathcal{H}} \otimes 1_{A} \tag{A.8}$$

$$S(h \otimes a) = (1_{\mathcal{H}} \otimes S_{\mathcal{A}}(h^{(1)}a))(S_H(h^{(2)}) \otimes 1_{\mathcal{A}}). \tag{A.9}$$

Let A, H be the duals of $\mathcal{A}, \mathcal{H}$. The dual 'group' aspect of the above formulae imply the existence of mappings $\bar{\alpha} : A \otimes H \to H, \bar{\beta} : A \to A \otimes H$, dual to (β, α) respectively, which satisfy the conditions

$$\varepsilon_H(a \,\bar{\triangleright}\, h) = \varepsilon_A(a)\varepsilon_H(h) \tag{A$'$.1}$$

$$\Delta(a \,\bar{\triangleright}\, h) \equiv (a \,\bar{\triangleright}\, h)_{(1)} \otimes (a \,\bar{\triangleright}\, h)_{(2)} = (a_{(1)}^{(1)} \,\bar{\triangleright}\, h_{(1)}) \otimes a_{(1)}^{(2)}(a_{(2)} \,\bar{\triangleright}\, h_{(2)}) \tag{A$'$.2}$$

$$\bar{\beta}(1_A) \equiv 1_A^{(1)} \otimes 1_A^{(2)} = 1_A \otimes 1_H \tag{A$'$.3}$$

$$\bar{\beta}(ab) \equiv (ab)^{(1)} \otimes (ab)^{(2)} = a_{(1)}^{(1)}b^{(1)} \otimes a_{(1)}^{(2)}(a_{(2)} \,\bar{\triangleright}\, b^{(2)}) \tag{A$'$.4}$$

$$a_{(2)}^{(1)} \otimes (a_{(1)} \,\bar{\triangleright}\, h)a_{(2)}^{(2)} = a_{(1)}^{(1)} \otimes a_{(1)}^{(2)}(a_{(2)} \,\bar{\triangleright}\, h). \tag{A$'$.5}$$

Then, there is a left–right bicrossproduct structure $H_{\bar{\alpha}} \blacktriangleright\!\!\triangleleft^{\bar{\beta}} A$ on $K = H \otimes A$ defined by

$$(h \otimes a)(g \otimes b) = h(a_{(1)} \,\bar{\triangleright}\, g) \otimes a_{(2)}b \qquad h, g \in H \ a, b \in A \tag{A$'$.6}$$

$$\Delta_K(h \otimes a) = h_{(1)} \otimes a_{(1)}^{(1)} \otimes h_{(2)}a_{(1)}^{(2)} \otimes a_{(2)} \tag{A$'$.7}$$

$$\varepsilon_K = \varepsilon_H \otimes \varepsilon_A \qquad 1_K = 1_H \otimes 1_A \tag{A$'$.8}$$

$$S(h \otimes a) = (1_H \otimes S_A(a^{(1)}))(S_H(ha^{(2)}) \otimes 1_A). \tag{A$'$.9}$$

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